

# Some New Bounds For Cover-Free Families Through Biclique Cover

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## Abstract

An  $(r, w; d)$  cover-free family (CFF) is a family of subsets of a finite set such that the intersection of any  $r$  members of the family contains at least  $d$  elements that are not in the union of any other  $w$  members. The minimum number of elements for which there exists an  $(r, w; d)$  – CFF with  $t$  blocks is denoted by  $N((r, w; d), t)$ .

In this paper, we show that the value of  $N((r, w; d), t)$  is equal to the  $d$ -biclique covering number of the bipartite graph  $I_t(r, w)$  whose vertices are all  $w$ - and  $r$ -subsets of a  $t$ -element set, where a  $w$ -subset is adjacent to an  $r$ -subset if their intersection is empty. Next, we introduce some new bounds for  $N((r, w; d), t)$ . For instance, we show that for  $r \geq w$  and  $r \geq 2$

$$N((r, w; 1), t) \geq c \frac{\binom{r+w}{w+1} + \binom{r+w-1}{w+1} + 3\binom{r+w-4}{w-2}}{\log r} \log(t - w + 1),$$

where  $c$  is a constant satisfies the well-known bound  $N((r, 1; 1), t) \geq c \frac{r^2}{\log r} \log t$ . Also, we determine the exact value of  $N((r, w; d), t)$  for some values of  $d$ . Finally, we show that  $N((1, 1; d), 4d - 1) = 4d - 1$  whenever there exists a Hadamard matrix of order  $4d$ .

**Key words:** cover-free family, biclique cover, fractional biclique cover, weakly cross-intersecting set-pairs.

**Subject classification:** 05B40.

## 1 Introduction

A family of sets is called an  $(r, w)$ -cover-free family if no intersection of  $r$  sets of the family are covered by a union of any other  $w$  sets of the family. Cover-free family was first introduced by Kautz and Singleton [17] to investigate the properties of the non-random binary superimposed codes. In 1985, Erdős, Frankl, and Füredi [14] studied

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this concept as a generalization of Sperner system. In 1988, Mitchell and Piper [22] defined the concept of key distribution pattern which is in fact a generalized type of cover-free family. Others have used this concept in cryptography, for example, group key predistribution, frameproof codes, broad cast anti-jamming, and so on, see [9]. Cover-free family has been studied extensively throughout the literature due to both its interesting structure and the central role it plays in several respects, see [5, 11, 14, 17, 30, 33].

In this paper, we discuss aspects relevant to cover-free families. In Section 1, we set up notation and terminology. Section 2 is devoted to study the connection between cover-free families and biclique cover. In Section 3, we presents several new lower bounds for  $N((r, w; d), t)$ . Finally, Section 4 concerns the fractional version of biclique cover and we determine the exact value of  $N((r, w; d), t)$  for some values of  $d$ . Finally, we show that if there exists a Hadamard matrix of order  $4d$ , then  $N((1, 1; d), 4d - 1) = 4d - 1$ .

Throughout this paper, we only consider finite simple graphs. For a graph  $G$ , let  $V(G)$  and  $E(G)$  denote its vertex and edge sets, respectively. By a *biclique* we mean a bipartite graph with vertex set  $(X, Y)$  such that every vertex in  $X$  is adjacent to every vertex in  $Y$ . Note that every empty graph is a biclique. A *biclique cover* of a graph  $G$  is a collection of bicliques of  $G$  such that each edge of  $G$  is in at least one of the bicliques. The number of bicliques in a minimum biclique cover of  $G$  is called the *biclique covering number* of  $G$  and denoted by  $bc(G)$ . This measure of graphs is studied in the literature [1, 3, 15].

In this paper, we also need a generalization of biclique cover as follows.

**Definition 1.** A *d-biclique cover* of a graph  $G$  is a collection of bicliques of  $G$  such that each edge of  $G$  is in at least  $d$  of the bicliques. The number of bicliques in a minimum *d-biclique cover* of  $G$  is called the *d-biclique covering number* of  $G$  and denoted by  $bc_d(G)$ . ♠

As usual, we denote by  $[t]$  the set  $\{1, 2, \dots, t\}$ . In this paper, by  $A^c$  we mean the complement of the set  $A$ . For  $0 < w \leq r \leq t$ , the *subset graph*  $S_t(w, r)$  is a bipartite graph whose vertices are all  $w$ - and  $r$ -subsets of a  $t$ -element set, where a  $w$ -subset is adjacent to an  $r$ -subset if and only if one subset is contained in the other. Some properties of this family of graphs have been studied by several researchers, see [12, 23, 27]. In this paper, we consider an isomorphic version of this graph and name it *bi-intersection graph*.

**Definition 2.** For  $0 < w \leq r \leq t$ , the *bi-intersection graph*  $I_t(r, w)$  is a bipartite graph whose vertices are all  $w$ - and  $r$ -subsets of a  $t$ -element set, where a  $w$ -subset is adjacent to an  $r$ -subset if and only if their intersection is empty. ♠

It is not difficult to see that the bi-intersection graph  $I_t(r, w)$  is isomorphic to  $S_t(r, t - w)$ . A *set system* is an ordered pair  $(X, \mathcal{B})$ , where  $X$  is a set of elements and  $\mathcal{B}$  is a family of subsets (called block) of  $X$ . A set system can be described by an incidence matrix. Let  $(X, \mathcal{B})$  be a set system, where  $X = \{x_1, x_2, \dots, x_v\}$  and  $\mathcal{B} = \{B_1, B_2, \dots, B_b\}$ . The incidence matrix of  $(X, \mathcal{B})$  is the  $b \times v$  matrix  $A = (a_{ij})$ , where

$$a_{ij} = \begin{cases} 1 & \text{if } x_j \in B_i \\ 0 & \text{if } x_j \notin B_i \end{cases}$$

The next definition is a formal definition of cover-free family.

**Definition 3.** Let  $n, t, r$ , and  $w$  be positive integers. A set system  $(X, \mathcal{B})$ , where  $|X| = n$  and  $\mathcal{B} = \{B_1, \dots, B_t\}$  is called an  $(r, w) - CFF(n, t)$  if for any two sets of indices  $L, M \subseteq [t]$  such that  $L \cap M = \emptyset$ ,  $|L| = r$ , and  $|M| = w$ , we have

$$\bigcap_{l \in L} B_l \not\subseteq \bigcup_{m \in M} B_m.$$

♠

Stinson and Wei [29] generalized the definition of cover-free family as follows.

**Definition 4.** Let  $d, n, t, r$ , and  $w$  be positive integers. A set system  $(X, \mathcal{B})$ , where  $|X| = n$  and  $\mathcal{B} = \{B_1, \dots, B_t\}$  is called an  $(r, w; d) - CFF(n, t)$  if for any two sets of indices  $L, M \subseteq [t]$  such that  $L \cap M = \emptyset$ ,  $|L| = r$ , and  $|M| = w$ , we have

$$|(\bigcap_{l \in L} B_l) \setminus (\bigcup_{m \in M} B_m)| \geq d.$$

♠

Let  $N((r, w; d), t)$  denote the minimum number of elements in any  $(r, w; d) - CFF$  having  $t$  blocks. For convenience, we use the notation  $N((r, w), t)$  instead of  $N((r, w; 1), t)$ . Obviously, we have  $N((r, w; d), t) = N((w, r; d), t)$ . Hence, unless otherwise stated we assume that  $w \leq r$ .

## 2 Biclique Cover

In this section, we show that the existence of a cover-free family can result from the existence of biclique cover of bi-intersection graph and vice versa. Our viewpoint sheds some new light on cover-free family. Using this observation, we introduce several new bounds.

**Theorem 1.** Let  $r, w$ , and  $t$  be positive integers, where  $t \geq r + w$ . It holds that

$$N((r, w), t) = bc(I_t(r, w)).$$

**Proof.** First, consider an optimal  $(r, w) - CFF(n, t)$ , i.e.,  $n = N((r, w), t)$ , with incidence matrix  $A = (a_{ij})$ . Assign to the  $j$ th column of  $A$ , the set  $A_j$  as follows

$$A_j \stackrel{\text{def}}{=} \{i \mid 1 \leq i \leq t, a_{ij} = 1\}.$$

Now, for any  $1 \leq j \leq n$ , construct a bipartite graph  $G_j$  with vertex set  $(X_j, Y_j)$ , where the vertices of  $X_j$  are all  $r$ -subsets of  $A_j$  and the vertices of  $Y_j$  are all  $w$ -subsets of  $A_j^c$ , i.e.,

$$X_j = \{U \mid U \subseteq A_j, |U| = r\} \quad \text{and} \quad Y_j = \{V \mid V \subseteq A_j^c, |V| = w\}.$$

Also, an  $r$ -subset is adjacent to a  $w$ -subset if their intersection is empty. One can see that  $G_j$ , for  $1 \leq j \leq n$ , is a biclique. Let  $UV$  be an arbitrary edge of  $I_t(r, w)$ , where  $U \cap V = \emptyset$ ,  $|U| = r$  and  $|V| = w$ . In view of definition of CFF and since  $A$  is the incidence matrix of the CFF, there is a column of  $A$ , say  $j$ , where  $a_{ij} = 1$  if  $i \in U$  and  $a_{ij} = 0$  if  $i \in V$ . Clearly,  $U \in X_j$ ,  $V \in Y_j$ , and  $UV \in G_j$ . Hence,  $\{G_1, G_2, \dots, G_n\}$  is a biclique cover of  $I_t(r, w)$ . So  $bc(I_t(r, w)) \leq N((r, w), t)$ .

Conversely, assume that  $G_1, \dots, G_l$  constitute a biclique cover of  $I_t(r, w)$ , where  $l = bc(I_t(r, w))$  and  $G_i$  has as its vertex set  $(X_i, Y_i)$ . Let  $A_i$  be the union of sets that lie in  $X_i$ . Consider the indicator vector of the set  $A_i$ , for  $i = 1, \dots, l$ , and construct the matrix  $A$  whose  $i$ th column is the indicator vector of the set  $A_i$ . We claim that  $A$  is the incidence matrix of an  $(r, w) - CFF(l, t)$ . To see this, let  $U$  and  $V$  be two arbitrary disjoint sets of  $[t]$ , where  $|U| = r$  and  $|V| = w$ . Thus,  $UV$  is an edge of the graph  $I_t(r, w)$ . Hence, there exists a biclique  $G_j$ , where  $U \in X_j$  and  $V \in Y_j$ . Now, in view of definition of  $A_j$ , one can see that all entries corresponding to the elements of  $U$  and  $V$  in the  $j$ th column are 1 and 0, respectively. So  $N((r, w), t) \leq bc(I_t(r, w))$ . This completes the proof.  $\blacksquare$

By the same argument we obtain the following corollary.

**Corollary 1.** *Let  $r, w, d$ , and  $t$  be positive integers, where  $t \geq r + w$ . It holds that*

$$N((r, w; d), t) = bc_d(I_t(r, w)).$$

A *weakly separating system* on  $[t]$  is a collection  $\{(X_1, Y_1), \dots, (X_n, Y_n)\}$  of disjoint pairs of subsets of  $[t]$  such that for every  $i, j \in [t]$  with  $i \neq j$  there is a  $k$  with either  $i \in X_k$  and  $j \in Y_k$  or  $i \in Y_k$  and  $j \in X_k$ . Similarly, a *strongly separating system* on  $[t]$  is a collection  $\{(X_1, Y_1), \dots, (X_n, Y_n)\}$  of disjoint pairs of subsets of  $[t]$  such that for every ordered pair  $(i, j)$  with  $1 \leq i, j \leq t$  and  $i \neq j$ , there is a  $k \in [n]$  with  $i \in X_k$  and  $j \in Y_k$ . The study of separating systems was started by Rényi [25] in 1961. Other researchers have studied the properties of separating system in the literature, see [8, 7, 28, 24]. One can construct a  $(1, 1) - CFF(n, t)$  from a strongly separating system on  $[t]$  of size  $n$  and vice versa (see the proof of Theorem 1). So if we denote by  $\mathcal{R}(t)$ , the minimum size of a strongly separating system, we have  $N((1, 1), t) = \mathcal{R}(t)$ . Let  $\{(X_1, Y_1), \dots, (X_n, Y_n)\}$  be a weakly separating system. The complete bipartite graphs with vertex classes  $X_i$  and  $Y_i$  cover the edges of the complete graph  $K_t$  with vertex set  $[t]$ . Also, if the family  $\{G_1, \dots, G_n\}$  is a biclique cover of  $K_t$ , where  $G_i$  has as its vertex set  $(X_i, Y_i)$ , then  $\{(X_1, Y_1), \dots, (X_n, Y_n)\}$  is a weakly separating system. So if we denote by  $s(t)$ , the size of minimum weakly separating system, then we have  $s(t) = bc(K_t)$ . Also, in [3], it was proved that  $\mathcal{R}(t) = bc(K_{t,t}^-)$ , where  $K_{t,t}^-$  is the complete bipartite graph  $K_{t,t}$  with a perfect matching removed. The exact value of  $\mathcal{R}(t)$  was determined by Spencer [28].

**Theorem A.** [28] *If  $C = \min\{c \mid \binom{c}{\lfloor \frac{c}{2} \rfloor} \geq t\}$ , then  $C = \mathcal{R}(t)$ .*

Theorem A implies

$$\mathcal{R}(t) = \log_2 t + \frac{1}{2} \log_2 \log_2 t + O(1).$$

It is simple to see that  $bc(G) \geq m(G)$ , where  $m(G)$  is the maximum size of induced matchings of  $G$ . Let  $\mathcal{F} = \{(A_i, B_i)\}_{i=1}^h$  be a family of pairs of subsets of an arbitrary set. The family  $\mathcal{F}$  is called an  $(r, w)$ -system if for all  $1 \leq i \leq h$ ,  $|A_i| = r$ ,  $|B_i| = w$ ,  $A_i \cap B_i = \emptyset$ , and for all distinct  $i, j$  with  $1 \leq i, j \leq h$ ,  $A_i \cap B_j \neq \emptyset$ . Bollobás [6] proved that the maximum size of an  $(r, w)$ -system is equal to  $\binom{r+w}{r}$ . Obviously,  $m(I_t(r, w))$  is the maximum size of an  $(r, w)$ -system, so  $N((r, w), t) \geq \binom{r+w}{r}$ . A *covering* of a graph  $G$  is a subset  $K$  of  $V(G)$  such that every edge of  $G$  has at least one end in  $K$ . The number of vertices in a minimum covering of  $G$  is called the *covering number* of  $G$  and denoted by  $\beta(G)$ . It is not difficult to see that the biclique covering number of a graph  $G$  without  $C_4$  as a subgraph is equal to the covering number of  $G$ . So we have the following corollary.

**Corollary 2.** *For any positive integers  $r, w$ , and  $t$ , where  $t = r + w + 1$  or  $t = r + w$ , we have*

$$N((r, w), t) = \min\left\{\binom{t}{r}, \binom{t}{w}\right\} = \binom{t}{w}.$$

**Proof.** It is easy to see that the graph  $I_t(r, w)$ , when  $t = r + w + 1$  or  $t = r + w$ , does not contain  $C_4$  as a subgraph. On the other hand, it is well-known that for every bipartite graph, the covering number is equal to the maximum size of matchings. Easily, using Hall's Theorem, the maximum number of matching in this graph is equal to  $\min\left\{\binom{t}{r}, \binom{t}{w}\right\}$ . ■

We should mention that it is known [11] that  $N((r, w), t) = \binom{t}{w}$  whenever  $t \leq r + w + \frac{r}{w}$ . As an interesting application of cover-free family, one can consider key predistribution scheme (KPS). The specification structure of a KPS is the family of all disjoint pairs  $(P, F)$  of subsets of the set of users  $U$  such that every user in  $P$  must be able to compute a common key of  $P$  that will remain unknown to the coalition  $F$ . The above corollary gives the exact value of the minimum number of the keys in a KPS, constructed by a cover-free family, with  $r + w + 1$  users.

### 3 Bounds

In this section, we introduce several bounds for  $N((r, w; d), t)$ . Engel [11], using the fractional matching and fractional cover of ordered interval hypergraph, obtained the following bounds

**Theorem B.** [11] *For any positive integers  $r, w$ , and  $t$ , where  $r \geq w$  and  $t \geq r + w$ , we have*

$$N((r, w), t) \geq \binom{r + w - 1}{r} \mathcal{R}(t - r - w + 2).$$

**Theorem C.** [11] *For any  $\epsilon > 0$ , it holds that*

$$N((r, w), t_\epsilon) \geq (1 - \epsilon) \frac{(w + r - 2)^{w+r-2}}{(w-1)^{w-1}(r-1)^{r-1}} \mathcal{R}(t_\epsilon - r - w + 2),$$

for all sufficiently large  $t_\epsilon$ .

Here is the best known lower bound for  $N((r, 1), t)$ .

**Theorem D.** [10, 16, 26] *Let  $r \geq 2$  and  $t \geq r + 1$  be positive integers. It holds that*

$$N((r, 1), t) \geq C_{r,t} \frac{r^2}{\log r} \log t,$$

where  $\lim_{r+t \rightarrow \infty} C_{r,t} = c$  for some constant  $c$ .

Several proofs have been presented for the preceding theorem. In [10, 16, 26], it was shown that  $c$  is approximately  $\frac{1}{2}$ ,  $\frac{1}{4}$ , and  $\frac{1}{8}$ , respectively.

**Lemma A.** [30] *For any positive integers  $r, w$ , and  $t$ , where  $t \geq r + w$ , we have*

$$N((r, w), t) \geq N((r, w-1), t-1) + N((r-1, w), t-1).$$

Stinson, Wei, and Zhu [30], using Lemma A and Theorem D, improved the bounds of Engel in some cases and obtained the following bounds.

**Theorem E.** [30] *For any positive integers  $r, w$ , and  $t$ , where  $t \geq r + w$ , we have*

$$N((r, w), t) \geq 2c \frac{\binom{w+r}{r}}{\log(w+r)} \log t,$$

where  $c$  is a constant satisfies Theorem D.

**Theorem F.** [30] *For any positive integers  $r, w \geq 1$ , there exists an integer  $t_{r,w}$  such that for all  $t > t_{r,w}$*

$$N((r, w), t) \geq 0.7c(r+w) \frac{\binom{w+r}{r}}{\log\left(\binom{w+r}{r}\right)} \log t,$$

where  $c$  is a constant satisfies Theorem D.

In [20], it was shown  $t_{r,w} \leq \max\{\lfloor \frac{r+w+1}{2} \rfloor^2, 5\}$ . Here we introduce some new lower bounds for  $N((r, w; d), t)$  which improve Theorem B and also we present a lower bound (Theorem 3) which can be considered as an improvement of Theorems E and F whenever  $w$  is sufficiently small relative to  $r$ . We first prove the following preliminary lemma which will be needed in the proof of Theorem 2.

**Lemma 1.** *Let  $G$  be a graph and  $G_1, G_2, \dots, G_k$  be some pairwise vertex disjoint subgraphs of  $G$ . Also, assume that for every four cycle  $C_4$  of  $G$  and  $1 \leq i \neq j \leq k$ , we have  $E(C_4) \cap E(G_i) = \emptyset$  or  $E(C_4) \cap E(G_j) = \emptyset$ . Then*

$$bc_d(G) \geq \sum_{i=1}^k bc_d(G_i).$$

**Proof.** Let  $\{H_1, H_2, \dots, H_l\}$  be an optimal  $d$ -biclique cover of  $G$ , i.e.,  $l = bc_d(G)$ . Let  $H'_i$  be a subgraph of  $G_1 \cup G_2 \cup \dots \cup G_k$  induced by  $H_i$ . If  $H'_i$  is a non-empty graph, by the assumption, it is clear that  $H'_i$  is a biclique of exactly one of  $G_i$ 's. Now one can see that  $H'_j$ 's cover all edges of  $G_i$ 's at least  $d$  times. So  $bc_d(G) \geq \sum_{i=1}^k bc_d(G_i)$ , as desired. ■

Before embarking on the proof of the next theorem, we need the following definition. The family  $\mathcal{F} = \{(A_1, B_1), \dots, (A_g, B_g)\}$  is called a *weakly cross-intersecting set-pairs* (resp. cross-intersecting set-pairs) of size  $g$  on a ground set of cardinality  $h$  whenever all  $A_i$ 's and  $B_i$ 's are subsets of an  $h$ -set and for every  $i$ , where  $1 \leq i \leq g$ ,  $A_i$  and  $B_i$  are disjoint subsets, and furthermore, for every  $i \neq j$ ,  $(A_i \cap B_j) \cup (A_j \cap B_i) \neq \emptyset$  (resp.  $(A_i \cap B_j) \neq \emptyset$  and  $(A_j \cap B_i) \neq \emptyset$ ). This concept is a variant of the generalization of  $(r, w)$ -weakly cross-intersecting set-pairs which was introduced first by Tuza [32]. The weakly cross-intersecting set-pairs  $\mathcal{F} = \{(A_1, B_1), \dots, (A_g, B_g)\}$  is called an  $(r, w)$ -*weakly cross-intersecting set-pairs* whenever for every  $1 \leq i \leq g$ ,  $|A_i| = r$  and  $|B_i| = w$ .

**Remark 1.** It is worth noting that one can consider any weakly cross-intersecting set-pairs as a biclique covering. To see this, assume that  $\{G_1, \dots, G_l\}$  is a biclique cover of a graph  $G$  and  $G_i$  has  $(X_i, Y_i)$ , as its vertex set. This biclique cover is called an  $(r, w)$ -biclique cover whenever each vertex of  $G$  belongs to at most  $r$  sets in  $\{X_1, X_2, \dots, X_l\}$  and at most  $w$  sets in  $\{Y_1, Y_2, \dots, Y_l\}$ . Let  $\{(A_1, B_1), \dots, (A_g, B_g)\}$  be a set-pairs on a ground set of size  $h$ . The dual set system  $\{(S_1, T_1), \dots, (S_h, T_h)\}$  is a set-pairs on a ground set of size  $g$  and is defined by  $S_i = \{j : i \in A_j\}$  and  $T_i = \{j : i \in B_j\}$ , for  $i = 1, 2, \dots, h$ . It is shown in [7] that a family is a cross-intersecting set-pairs if and only if its dual is a strongly separating system. Similarly, one can see that  $\{(A_1, B_1), \dots, (A_g, B_g)\}$  is a weakly cross-intersecting set-pairs on a ground set of size  $h$  such that for any  $1 \leq i \leq g$ ,  $|A_i| \leq r$  and  $|B_i| \leq w$  if and only if the dual of this set-pairs system, i.e.  $\{(S_1, T_1), \dots, (S_h, T_h)\}$ , is an  $(r, w)$ -biclique cover of size  $h$  for the complete graph  $K_g$ .

Hereafter, we adopt the convention that  $N((r, 0; d), t) = N((0, w; d), t) = 1$ .

**Theorem 2.** Suppose that  $g, h, r, w$ , and  $t$  are positive integers. Also, let  $\mathcal{F} = \{(A_1, B_1), \dots, (A_g, B_g)\}$  be a weakly cross-intersecting set-pairs on a ground set of size  $h$  such that for any  $1 \leq i \leq g$ ,  $|A_i| \leq r$  and  $|B_i| \leq w$ . If  $t \geq \max\{h, r + w\}$ , then

$$N((r, w; d), t) \geq \sum_{i=1}^g N((r - |A_i|, w - |B_i|; d), t - |A_i| - |B_i|).$$

**Proof.** Assume that  $\mathcal{F} = \{(A_1, B_1), \dots, (A_g, B_g)\}$  is a weakly cross-intersecting set-pairs. For every  $1 \leq k \leq g$ , construct a bipartite graph  $G_k$  with vertex set  $(X_k, Y_k)$ , where the vertices of  $X_k$  are all  $r$ -subsets of  $[t]$  which contain  $A_k$  and their intersections with  $B_k$  are empty. Also, the vertices of  $Y_k$  are all  $w$ -subsets of the set  $[t]$  which contain  $B_k$  and their intersections with  $A_k$  are empty, i.e.,

$$X_k = \{U \mid U \subseteq [t], |U| = r, A_k \subseteq U, U \cap B_k = \emptyset\}$$

$$Y_k = \{V \mid V \subseteq [t], |V| = w, B_k \subseteq V, V \cap A_k = \emptyset\},$$

where a vertex  $U \in X_k$  is adjacent to a vertex  $V \in Y_k$  if  $U \cap V = \emptyset$ . Clearly, if  $|A_k| = r$  or  $|B_k| = w$ , then  $G_k$  is isomorphic to a star graph. Otherwise, one can check that every  $G_k$  is isomorphic to  $I_{t-|A_k|-|B_k|}(r-|A_k|, w-|B_k|)$ . Since if we delete the elements of  $A_k$  from the vertices of  $X_k$ , every vertex is mapped to an  $(r-|A_k|)$ -subset



of the set  $[t] \setminus (A_k \cup B_k)$  and also if we remove the elements of  $B_k$  from the vertices of  $Y_k$ , every vertex is mapped to a  $(w - |B_k|)$ -subset of the set  $[t] \setminus (A_k \cup B_k)$ . Obviously, this mapping is an isomorphism between  $G_k$  and  $I_{t-|A_k|-|B_k|}(r - |A_k|, w - |B_k|)$ . On the other hand, since  $\mathcal{F}$  is a weakly cross-intersecting set-pairs,  $G_k$ 's are pairwise vertex disjoint. Also, for any  $1 \leq i \neq j \leq k$ , there is no four cycle  $C_4$  of  $I_t(r, w)$  such that  $E(C_4) \cap E(G_i) \neq \emptyset$  and  $E(C_4) \cap E(G_j) \neq \emptyset$ . So, in view of Lemma 1, one can see that

$$bc_d(I_t(r, w)) \geq \sum_{k=1}^h bc_d(G_k).$$

Hence, the result easily follows.  $\blacksquare$

Here, we mention some consequences of the above theorem. Let  $M$  be an  $s$ -subset of  $[t]$ . For any non-negative integers  $i$  and  $j$ , where  $s - w \leq i \leq r$  and  $s - r \leq j \leq w$ , set

$$\begin{aligned} \mathcal{F}_i &= \{(A^i, B^i) : A^i \subseteq M, |A^i| = i, B^i = M \setminus A^i\}, \\ \mathcal{E}_j &= \{(A^j, B^j) : A^j \subseteq M, |A^j| = j, B^j = M \setminus A^j\}. \end{aligned}$$

It is easy to see that  $|\mathcal{F}_i| = \binom{s}{i}$  and  $|\mathcal{E}_j| = \binom{s}{j}$ . Also, it is not difficult to see that  $\mathcal{F} = \cup_{s-w \leq i \leq r} \mathcal{F}_i$  (resp.  $\mathcal{E} = \cup_{s-r \leq j \leq w} \mathcal{E}_j$ ) is a weakly cross-intersecting set-pairs. Therefore, in view of Theorem 2, we have the following corollary which is a generalization of Lemma A (set  $s = 1$ ).

**Corollary 3.** *For any positive integers  $0 < s \leq r + w$  and  $t \geq r + w$ , it holds that*

$$\begin{aligned} 1. \quad N((r, w; d), t) &\geq \sum_{s-w \leq i \leq r} \binom{s}{i} N((r - i, w - s + i; d), t - s), \\ 2. \quad N((r, w; d), t) &\geq \sum_{s-r \leq j \leq w} \binom{s}{j} N((r - s + j, w - j; d), t - s). \end{aligned}$$

Let  $T((r, w); n)$  denote the maximum number of blocks in an  $(r, w)$ -CFF with  $n$  points. Erdős et al. [13] discussed  $(1, 2)$ -CFFs in detail, and showed that

$$1.134^n \leq T((1, 2); n) \leq 1.25^n.$$

The upper bound is asymptotic and for sufficiently large  $n$  is useful. Hence, for large  $n$ , we have  $N((1, 2); t) \geq \frac{1}{\log(1.25)} \log t$ . If we set  $s = r + w - 3$  in the above corollary, then the following bound can be concluded which can be considered as an improvement of Theorem B.

**Corollary 4.** *For any positive integers  $r$  and  $w$ , where  $r \geq 2$ , it holds that*

$$N((r, w), t) \geq \binom{r + w - 2}{r - 1} N((2, 1); t - r - w + 3) + \binom{r + w - 3}{r} + \binom{r + w - 3}{r - 3}.$$

In view of Theorem 2, if there exists an  $(i, j)$ -weakly cross-intersecting set-pairs, then the following corollary can be concluded. We should mention that Engel [11] obtained a result that is similar to the following corollary.



**Corollary 5.** *Let  $i, j, r$ , and  $w$  be positive integers, where  $1 \leq i \leq r-1$  and  $1 \leq j \leq w-1$ . If there exists an  $(i, j)$ -weakly cross-intersecting set-pairs of size  $g(i, j)$  on a ground set of cardinality  $h$ , then for any  $t$ , where  $t \geq \max\{h, r+w\}$ , we have*

$$N((r, w; d), t) \geq g(i, j)N((r-i, w-j; d), t-i-j).$$

By a lattice path we mean a path on an  $i \times j$  grid from  $(0, 0)$  to  $(i, j)$ , where each move is to the right or up. Assume that  $\mathcal{L}(i, j)$  is the set of lattice paths such that the path is strictly below the line  $y = \frac{j}{i}x$  except at the two endpoints. Tuza [32] showed that if  $f(i, j)$  is the maximum size of a weakly cross-intersecting set-pairs, then  $f(i, j) < \frac{(i+j)^{i+j}}{i^i j^j}$ . Recently, Z. Király, Z.L. Nagy, D. Pálvolgyi, and M. Visontai [18], by a charming idea and using lattice paths, presented an  $(i, j)$ -weakly cross-intersecting set-pairs of size  $(2i+2j-1)|\mathcal{L}(i, j)|$  on a ground set of size  $2i+2j-1$ . Unfortunately, for general  $(i, j)$ , there is no explicit formula for  $|\mathcal{L}(i, j)|$ . However, Bizley [4] showed that for relatively prime numbers  $i$  and  $j$ ,  $|\mathcal{L}(i, j)| = \frac{(i+j)}{i+j}$ . In [18], it is shown  $g(i, j) \geq (2 - o(1))\binom{i+j}{i}$ , where  $f \in o(1)$  means that  $\lim_{i+j \rightarrow \infty} f = 0$ .

**Corollary 6.** *Assume that  $r, w$ , and  $t$  are positive integers, where  $t \geq \max\{2r+2w-5, r+w\}$ . Then*

$$N((r, w), t) \geq (2 - o(1))\binom{r+w-2}{r-1}\mathcal{R}(t-r-w+2).$$

Also, in [18], it is shown that there exists an  $(r-1, r-1)$ -weakly cross-intersecting set-pairs of size  $(2 - \frac{1}{2r-2})\binom{2r-2}{r-1}$  on a ground set of size  $4r-6$ .

**Corollary 7.** *Assume that  $r$  and  $t$  are positive integers, where  $t \geq \max\{4r-6, 2r\}$ . Then*

$$N((r, r), t) \geq (2 - \frac{1}{2r-2})\binom{2r-2}{r-1}\mathcal{R}(t-2r+2).$$

**Remark 2.** It is worth pointing out that the lattice problem is a special case of the generalized ballot problem. Suppose that in an election, candidate  $A$  receives  $r$  votes and candidate  $B$  receives  $w$  votes. Let  $r_i$  and  $w_i$  denote the number of votes  $A$  and  $B$  have after counting the  $i^{\text{th}}$  vote where  $1 \leq i \leq r+w$  (notice that  $r_i + w_i = i$ ). Let  $k$  be any positive real number. We call a sequence good if  $r > kw$  and  $r_i > kw_i$  for all  $r$ . We show the maximum number of good sequence by  $\mathcal{B}(r, w; k)$ . In 1887, Bertrand [2] showed  $\mathcal{B}(r, w; 1) = \frac{r-w}{r+w}\binom{r+w}{r}$ . Determining the exact value of  $\mathcal{B}(r, w; k)$  is known as the generalized ballot problem. It is not difficult to see that  $\mathcal{B}(r, w-1; \frac{r}{w}) = |\mathcal{L}(r, w)|$ . The solution to the generalized ballot problem when  $k$  is a positive integer is  $\frac{r-kw}{r+w}\binom{r+w}{r}$ . Unfortunately, for general  $k$ , there is no explicit formula for this problem. In 1962, Takacs [31] obtained a recurrence formula for the generalized ballot problem. Recently, Delong Meng [21] obtained a lower and upper bound for the generalized ballot problem.

The aforementioned bounds improve the existing bounds when the value of  $|r-w|$  is small. Now, we present another lower bound which is an improvement of the earlier bounds whenever  $w$  is sufficiently small relative to  $r$ . Moreover, this bound holds for any  $t \geq r+w$ .

**Theorem 3.** For any positive integers  $r$ ,  $w$ , and  $t$ , where  $t \geq r + w$ ,  $r \geq w$ , and  $r \geq 2$ , we have

$$N((r, w), t) \geq c \frac{\binom{r+w}{w+1} + \binom{r+w-1}{w+1} + 3\binom{r+w-4}{w-2}}{\log r} \log(t - w + 1),$$

where  $c$  is a constant satisfies Theorem D.

**Proof.** We prove the assertion by induction on  $w$ . By Theorem D, the assertion holds for  $w = 1$ . Assume that the assertion is true for every  $w'$  where  $w' < w$ . Easily, one can see that the family

$$\mathcal{F} = \{(\emptyset, \{1\}), (\{1\}, \{2\}), (\{1, 2\}, \{3\}), \dots, (\{1, 2, \dots, r-w\}, \{r-w+1\}), (\{1, 2, \dots, r-w+1\}, \{\emptyset\})\}$$

is a weakly cross-intersecting set-pairs. Hence, in view of Theorem 2, it holds that

$$\begin{aligned} N((r, w), t) &\geq (\sum_{i=0}^{r-w} N((r-i, w-1), t-i-1)) + N((w-1, w), t-r+w-1) \\ &= (\sum_{i=0}^{r-w} N((r-i, w-1), t-i-1)) + N((w, w-1), t-r+w-1) \end{aligned}$$

Now by induction we have

$$\begin{aligned} N((r, w), t) &\geq \sum_{i=0}^{r-w} c \frac{\binom{r+w-i-1}{w} + \binom{r+w-i-2}{w} + 3\binom{r+w-i-5}{w-3}}{\log(r-i)} \log(t - w + 1 - i) \\ &\quad + c \frac{\binom{2w-1}{w} + \binom{2w-2}{w} + 3\binom{2w-5}{w-3}}{\log(w)} \log(t - r + 1). \end{aligned}$$

Since  $\frac{\log x}{\log(x-1)}$  is a decreasing function, it holds that

$$\begin{aligned} N((r, w), t) &\geq c \frac{\log(t-w+1)}{\log r} (\sum_{i=0}^{r-w} \binom{r+w-i-1}{w} + \binom{r+w-i-2}{w} + 3\binom{r+w-i-5}{w-3}) \\ &\quad + c \frac{\log(t-w+1)}{\log r} (\binom{2w-1}{w} + \binom{2w-2}{w} + 3\binom{2w-5}{w-3}) \\ &\geq c \frac{\log(t-w+1)}{\log r} (\sum_{i=0}^{r-w} \binom{r+w-i-1}{w} + \binom{r+w-i-2}{w} + 3\binom{r+w-i-5}{w-3}) \\ &\quad + c \frac{\log(t-w+1)}{\log r} (\binom{2w-1}{w+1} + \binom{2w-2}{w+1} + 3\binom{2w-5}{w-2}) \\ &= c \frac{\binom{r+w}{w+1} + \binom{r+w-1}{w+1} + 3\binom{r+w-4}{w-2}}{\log r} \log(t - w + 1). \end{aligned}$$

■

## 4 Fractional Biclique Cover

The next result concerns the fractional version of biclique cover. If  $\mathcal{R}$  is the set of all bicliques of a graph  $G$ , then each biclique cover of  $G$  can be described by a function  $\phi : \mathcal{R} \rightarrow \{0, 1\}$  such that  $\phi(G_i) = 1$  if and only if  $G_i$  belongs to the cover. Hence,

$bc(G)$  is the minimum of  $\sum_{G_i \in \mathcal{R}} \phi(G_i)$  over all function  $\phi : \mathcal{R} \rightarrow \{0, 1\}$  such that for any edge  $e$  of  $G$  we have

$$\sum_{G_i \in \mathcal{R}: e \in E(G_i)} \phi(G_i) \geq 1. \quad (1)$$

The fractional biclique covering number  $bc^*(G)$  is the minimum of  $\sum_{G_i \in \mathcal{R}} \phi(G_i)$  over

all functions  $\phi : \mathcal{R} \rightarrow [0, 1]$  satisfying (1).

Fractional graph theory is the modification of integer-valued graph parameters to take its value on non-integer values. For more on fractional graph theory and other fractional graph parameters, see [27]. In the fractional cover, using linear programming, it is proved that

$$bc^*(G) = \inf_d \frac{bc_d(G)}{d} = \lim_{d \rightarrow \infty} \frac{bc_d(G)}{d}.$$

Also, we have the following theorem.

**Theorem G.** [27] *For every non-empty edge-transitive graph  $G$ , we have*

$$bc^*(G) = \frac{|E(G)|}{B(G)},$$

where  $B(G)$  is the maximum number of edges among the bicliques of  $G$ .

Easily, one can see that

$$B(I_t(r, w)) = \max_{t' + t'' = t} \binom{t'}{r} \binom{t''}{w}.$$

Also, we have  $|E(I_t(r, w))| = \binom{t}{r} \binom{t-r}{w}$ , and  $I_t(r, w)$  is an edge-transitive graph. Therefore, in view of Theorem G, we have

$$bc^*(I_t(r, w)) = \min_{t' + t'' = t} \frac{\binom{t}{r} \binom{t-r}{w}}{\binom{t'}{r} \binom{t''}{w}}.$$

By a straightforward calculation, one can see that

$$bc^*(I_t(r, w)) = \min_{t' + t'' = t} \frac{\binom{t}{r} \binom{t-r}{w}}{\binom{t'}{r} \binom{t''}{w}} = \min_{w \leq m \leq t-r} \frac{\binom{t}{m}}{\binom{t-r-w}{m-w}}.$$

Lovász [19] proved that for any graph  $G$  with maximum degree  $\Delta(G)$

$$bc^*(G) \geq \frac{bc(G)}{1 + \ln(\Delta(G))}.$$

The maximum degree of the graph  $I_t(r, w)$  is equal to

$$\max\left\{\binom{t-w}{r}, \binom{t-r}{w}\right\} = \binom{t-w}{r}.$$

So we have the following corollary.

**Corollary 8.** For any positive integers  $r$ ,  $w$ , and  $t$ , where  $t \geq r + w$ , we have

$$N((r, w), t) \leq \min_{w \leq m \leq t-r} \frac{\binom{t}{m}}{\binom{t-r-w}{m-w}} (1 + \ln\left(\binom{t-w}{r}\right)).$$

In [11], Engel proved that

$$N((r, w), t) \geq \min_{w-1 \leq m \leq t-r+1} \frac{\binom{t}{m}}{\binom{t-r-w+2}{m-w+1}} (N((1, 1), t-r-w+2)).$$

Hence, we have

$$N((r, w), t) \geq \min_{w-1 \leq m \leq t-r+1} \frac{\binom{t}{m}}{\binom{t-r-w+2}{m-w+1}} (\log_2(t-r-w+2) + \frac{1}{2} \log_2 \log(t-r-w+2) + c),$$

where  $c$  is a constant. In the next theorem, we specify the exact value of  $N((r, w; d), t)$  for some special value of  $d$ . In the proof of the next theorem, by  $S_t$  we mean the permutation group of the set  $[t]$ .

**Theorem 4.** For any positive integers  $r$ ,  $w$ ,  $t$ ,  $d_0$ , and  $d = \frac{B(I_t(r, w))}{|E(I_t(r, w))|} t!$ , where  $t \geq r + w$ , we have

$$N((r, w; d_0 d), t) = d_0(t!).$$

**Proof.** For every  $\sigma \in S_t$ , define the function  $f_\sigma : V(I_t(r, w)) \rightarrow V(I_t(r, w))$  such that for every set  $A = \{i_1, i_2, \dots, i_l\} \in V(I_t(r, w))$ , we have  $f_\sigma(A) = \{\sigma(i_1), \dots, \sigma(i_l)\}$  (note that here  $l = r$  or  $l = w$ ). Set  $G = \{f_\sigma \mid \sigma \in S_t\}$ . One can see that  $G$  is a subgroup of  $\text{Aut}(I_t(r, w))$  and also  $G$  acts transitively on  $E(I_t(r, w))$ . Now it is simple to check that

$$\frac{bc_d(I_t(r, w))}{d} = \frac{|E(I_t(r, w))|}{B(I_t(r, w))}.$$

To see this, assume that  $K$  is a biclique of  $I_t(r, w)$ , where  $|E(K)| = B(I_t(r, w))$ . Construct a biclique cover of  $I_t(r, w)$  as follows. Set

$$\mathcal{C} = \{f_\sigma(K) \mid \sigma \in S_t\}.$$

It is readily seen that  $\mathcal{C}$  is a biclique cover and every edge is covered with exactly  $d = \frac{B(I_t(r, w))t!}{|E(I_t(r, w))|}$  bicliques. So

$$\frac{bc_d(I_t(r, w))}{d} \leq \frac{|E(I_t(r, w))|}{B(I_t(r, w))}.$$

On the other hand, by the definition of fractional biclique cover, for every graph  $G$  and every positive integer  $d$  we have  $bc^*(G) \leq \frac{bc_d(G)}{d}$ . Particularly,

$$bc^*(I_t(r, w)) \leq \frac{bc_d(I_t(r, w))}{d}.$$

Consequently, in view of Theorem G, we have

$$\frac{bc_d(I_t(r, w))}{d} = \frac{|E(I_t(r, w))|}{B(I_t(r, w))}.$$

Also, for any positive integer  $d_0$  we have

$$bc_{d_0 d}(I_t(r, w)) \leq d_0 bc_d(I_t(r, w)).$$

Hence,

$$\frac{|E(I_t(r, w))|}{B(I_t(r, w))} = bc^*(I_t(r, w)) \leq \frac{bc_{d_0 d}(I_t(r, w))}{d_0 d} \leq \frac{bc_d(I_t(r, w))}{d} = \frac{|E(I_t(r, w))|}{B(I_t(r, w))}. \quad (2)$$

Consequently, using (2) we obtain the result.  $\blacksquare$

**Corollary 9.** *For any positive integers  $r$ ,  $w$ , and  $c$ , we have*

$$N((r, w; c(r+1)!w!), r+w+1) = c(r+w+1)!.$$

**Proof.** Set  $G := I_{r+w+1}(r, w)$ . The graph  $G$  is  $C_4$  free. So  $B(G) = r+1$ . Also,  $|E(G)| = (w+1)\binom{r+w+1}{r}$ . Hence, by a straightforward calculation and using Theorem 4, the corollary follows.  $\blacksquare$

An  $n \times n$  matrix  $H$  with entries  $+1$  and  $-1$  is called a *Hadamard matrix* of order  $n$  if  $HH^t = nI$ . It is seen that any two distinct columns of  $H$  are orthogonal. Also, if we multiply some rows or columns by  $-1$ , or if we permute rows or columns, then  $H$  is still a Hadamard matrix. Two such Hadamard matrices are called equivalent. Easily, for any Hadamard matrix  $H$ , we can find an equivalent one for which the first row and the first column consist entirely of  $+1$ 's. Such a Hadamard matrix is called *normalized*.

**Theorem 5.** *Let  $d$  be a positive integer such that there exists a Hadamard matrix of order  $4d$ , then  $N((1, 1; d), 4d-1) = 4d-1$ .*

**Proof.** Let  $H$  be a normalized Hadamard matrix of order  $4d$ . Delete the first row and the first column. Also, assume that  $K_{4d-1, 4d-1}^-$  has  $(X, Y)$  as its vertex set where  $X = \{v_1, \dots, v_{4d-1}\}$  and  $Y = \{v'_1, \dots, v'_{4d-1}\}$ . Assign to the  $j$ th column of  $H$ , two sets  $X_j$  and  $Y_j$  as follow

$$X_j = \{v_i | h_{ij} = +1\} \quad \& \quad Y_j = \{v'_i | h_{ij} = -1\}.$$

Construct a complete bipartite graph  $G_j$  with vertex set  $(X_j, Y_j)$ . The edge  $v_i v'_j$  is covered by the complete bipartite graph  $G_k$  if and only if the corresponding entries of column  $k$  in row  $i$  is  $+1$  and in row  $j$  is  $-1$ . It is well-known that the number of these columns, in a normalized Hadamard matrix of order  $4d$ , is equal to  $d$ . Hence, every edge is covered exactly  $d$  times. So  $bc_d(K_{4d-1, 4d-1}^-) \leq 4d-1$ . On the other hand,  $K_{4d-1, 4d-1}^-$  is an edge-transitive graph. Therefore, in view of Theorem G, we have

$$\frac{4d-1}{d} = \frac{|E(K_{4d-1, 4d-1}^-)|}{B(K_{4d-1, 4d-1}^-)} \leq \frac{bc_d(K_{4d-1, 4d-1}^-)}{d}.$$

Consequently,  $4d-1 \leq bc_d(K_{4d-1, 4d-1}^-)$  and the result follows.  $\blacksquare$

## References

- [1] Noga Alon. Neighborly families of boxes and bipartite coverings. In *The mathematics of Paul Erdős, II*, volume 14 of *Algorithms Combin.*, pages 27–31. Springer, Berlin, 1997. [2](#)
- [2] J. Bertrand. Solution d’un problme. *Comptes Rendus de l’Académie des Sciences*, Paris 105:369, 1887. [9](#)
- [3] Sergei Bezrukov, Dalibor Fronček, Steven J. Rosenberg, and Petr Kovář. On biclique coverings. *Discrete Math.*, 308(2-3):319–323, 2008. [2](#), [4](#)
- [4] M. T. L. Bizley, Derivation of a new formula for the number of minimal lattice paths from  $(0; 0)$  to  $(km; kn)$  having just  $t$  contacts with the line  $my = nx$  and having no points above this line; and a proof of Grossman’s formula for the number of paths which may touch but do not rise above this line. *J. Inst. Actuar.*, 80:55-62, 1954. [9](#)
- [5] Simon R. Blackburn. Frameproof codes. *SIAM J. Discrete Math.*, 16(3):499–510 (electronic), 2003. [2](#)
- [6] B. Bollobás. On generalized graphs. *Acta Math. Acad. Sci. Hungar*, 16:447–452, 1965. [5](#)
- [7] Béla Bollobás and Alex Scott. On separating systems. *European J. Combin.*, 28(4):1068–1071, 2007. [4](#), [7](#)
- [8] Béla Bollobás and Alex Scott. Separating systems and oriented graphs of diameter two. *J. Combin. Theory Ser. B*, 97(2):193–203, 2007. [4](#)
- [9] Charles J. Colbourn and Jeffrey H. Dinitz, editors. *Handbook of combinatorial designs*. Discrete Mathematics and its Applications (Boca Raton). Chapman & Hall/CRC, Boca Raton, FL, second edition, 2007. [2](#)
- [10] A. G. D’yachkov and V. V. Rykov. Bounds on the length of disjunctive codes. *Problemy Peredachi Informatsii*, 18(3):7–13, 1982. [6](#)
- [11] Konrad Engel. Interval packing and covering in the Boolean lattice. *Combin. Probab. Comput.*, 5(4):373–384, 1996. [2](#), [5](#), [8](#), [12](#)
- [12] T. C. Enns. Hamiltonian circuits and paths in subset graphs with circular adjacency. *Discrete Math.*, 122(1-3):153–165, 1993. [2](#)
- [13] P. Erdős, P. Frankl, and Z. Füredi. Families of finite sets in which no set is covered by the union of two others. *J Combin Theory A*, 33 (1982), 158166. [8](#)
- [14] P. Erdős, P. Frankl, and Z. Füredi. Families of finite sets in which no set is covered by the union of  $r$  others. *Israel J. Math.*, 51(1-2):79–89, 1985. [1](#), [2](#)
- [15] P. Erdős and L. Pyber. Covering a graph by complete bipartite graphs. *Discrete Math.*, 170(1-3):249–251, 1997. [2](#)

- [16] Zoltán Füredi. On  $r$ -cover-free families. *J. Combin. Theory Ser. A*, 73(1):172–173, 1996. 6
- [17] W. H. Kautz and R. C. Singleton. Nonrandom binary superimposed codes. *IEEE Trans Inf Theory*, 10:363–377, 1964. 1, 2
- [18] Zoltán Király, Zoltán L. Nagy, Dömötör Pálvölgyi, Mirkó Visontai. On weakly intersecting pairs of sets. *manuscript*, 2010. 9
- [19] L. Lovász. On the ratio of optimal integral and fractional covers. *Discrete Math.*, 13(4):383–390, 1975. 11
- [20] X. Ma and R. Wei. On a Bound of Cover-Free Families. *Des. Codes Cryptogr*, 32:303–321, 2004. 6
- [21] Delong Meng. Nice Bounds for the Generalized Ballot Problem. *manuscript*, 2009. 9
- [22] Chris J. Mitchell and Fred C. Piper. Key storage in secure networks. *Discrete Appl. Math.*, 21(3):215–228, 1988. 2
- [23] Jennifer J. Quinn and Arthur T. Benjamin. Strong chromatic index of subset graphs. *J. Graph Theory*, 24(3):267–273, 1997. 2
- [24] Colin Ramsay and Ian T. Roberts. Minimal completely separating systems of sets. *Australas. J. Combin.*, 13:129–150, 1996. 4
- [25] A. Rényi. On random generating elements of a finite Boolean algebra. *Acta Sci. Math. Szeged*, 22:75–81, 1961. 4
- [26] Miklós Ruszinkó. On the upper bound of the size of the  $r$ -cover-free families. *J. Combin. Theory Ser. A*, 66(2):302–310, 1994. 6
- [27] Edward R. Scheinerman and Daniel H. Ullman. *Fractional graph theory*. Wiley-Interscience Series in Discrete Mathematics and Optimization. John Wiley & Sons Inc., New York, 1997. A rational approach to the theory of graphs, With a foreword by Claude Berge, A Wiley-Interscience Publication. 2, 11
- [28] Joel Spencer. Minimal completely separating systems. *J. Combinatorial Theory*, 8:446–447, 1970. 4
- [29] D. R. Stinson and R. Wei. Generalized cover-free families. *Discrete Math.*, 279(1-3):463–477, 2004. In honour of Zhu Lie. 3
- [30] D. R. Stinson, R. Wei, and L. Zhu. Some new bounds for cover-free families. *J. Combin. Theory Ser. A*, 90(1):224–234, 2000. 2, 6
- [31] L. Takacs. A generalization of the ballot problem and its application to the theory of queues. *J. Amer. Statist. Assoc*, 57:327–337, 1962. 9
- [32] Zs. Tuza. Inequalities for two set systems with prescribed intersections. *Graphs and Combinatorics*, 3 (1987), 75–80. 7, 9
- [33] R. Wei. On cover-free families. *manuscript*, 2006. 2